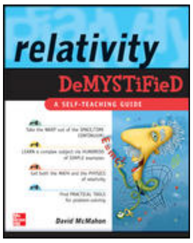

RELATIVITY DEMYSTIFIED ERRATA, COMMENTS & SOME QUESTION MARKS TOO...

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Foreword

I always wanted to learn about General Relativity which has a reputation of being one of the most beautiful physical theory, however I found the mathematics quite daunting when one (not so remote) day I find *"Relativity DeMYSTiFieD"*. I's a really very useful book which gives you what GR is about without skipping the math that the theory requires. Because I love this book, I wanted to correct some typos (I have not found any errata yet on the Internet). Because I have got some problems with some examples and quiz, I would like to share my experience with others to try to solve these difficulties. In addition I added somme comments to justify some statements without demonstration(don't wary just in some easy cases !). All feedback will be welcome (gandremarie [at] gmail [dot] com) and I will update the document accordingly. As you can guess I am not an expert in the field, I am just curious about the physical world we are living into.

I learn myself about tensors for the first time by Electro-Magnetism, therefore I will give some results for the EM field when I think it could help understand analogous calculations for the gravitational field.

Mathematical conventions

To save typing I will use when appropriate:
$A^a{}_{,b}$ for partial derivation instead of $\partial_b A^a$
$A^a{}_{;b}$ for covariant derivative instead of $\nabla_b A^a$

References

All mentioned page numbers and references in parenthesis refer to the book *"Relativity DeMYSTiFieD"*. References without parenthesis refer to sections of this paper.

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1. Chapter 1

Nothing really worth to report. (To be honest I rather skimmed over this chapter as I was eager to read the next ones).
However I cannot resist mentioning the Mawxell's equations in tensorial form which are then reduced to only two equations (to wet your appetite for tensors !)

$$e^{iklm}\partial_k F_{lm} = 0 \quad \text{which yields } \nabla \cdot B = 0 \quad \text{and the Maxwell-Faraday's law}$$

$$\text{(MKSA source equation) } \partial_k F^{ik} = -\frac{j^i}{c\epsilon_0} \quad \text{which yields (1.6) and (1.9)}$$

The F Faraday tensor being defined later in section 5.1

By identifying the Maxwell's equations mentioned in page 2 & 3 and the ones in tensorial form above, I have just realized that a key equation $\nabla \wedge e = -\frac{\partial B}{\partial t}$ or Maxwell-Faraday's law is missing in the book, therefore it is not only a question of simple vector calculus to retrieve the wave equations ! (To reassure the reader who would have tried to retrieve them). It's a pity that this first presentation was done in too much of a hurry.
Nevertheless I found the velocity composition law demonstration rather lenghty...
It is rather straitforwards to retreive it by elementary calculations. Starting by the Lorentz formulae :

$$ct' = \gamma\left(ct - x\frac{V_e}{c}\right),$$
$$x' = \gamma(x - V_e t)$$

with $\gamma = (1 - V_e^2/c^2)^{-1/2}$
Considering a point M moving with respect to R' following to the equation $x' = wt'$ and according to the Lorentz formulae above, one can express x' and t' function of x and t, and so :

$$\gamma(x - V_e t) = w\gamma(t - x\frac{V_e}{c^2})$$

which becomes :

$$x\left(1 + \frac{wV_e}{c^2}\right) = (V_e + w)t$$

and finally :

$$\frac{x}{t} = W = \frac{V_e + w}{1 + \frac{V_e w}{c^2}} \quad \text{QED}$$

2. Chapter 2

Not a lot, page 35 about formula (2.15) $g_{ab}g^{bc} = \delta^c_a$ it should be noted when $a = c$,

$$g_{ab}g^{ab} = \delta^a_a \quad = \text{trace of } \delta^a_a = 4 \quad \text{(in 4D)}$$

This result will be useful in the future.
Page 44, inside Solution 2-5 , ∂r should be :

$$\frac{\partial x}{\partial r}\partial e_x + \frac{\partial y}{\partial r}\partial e_y + \frac{\partial z}{\partial r}\partial e_z$$

3. Chapter 3

Nothing worth noticing.

4. Chapter 4

4.1. Page 67, equation (4.6) should be :

$$\nabla_a A^b = \frac{\partial A^b}{\partial x^a} + \Gamma^b_{ca} A^c$$

4.2. Page 69, equation (4.11), there is no demonstration which gives the covariant derivative of a one form or covariant vector. Here is a nice demonstration (from Landau -Tome II Section 85-). Let consider the scalar or dot product of two vectors (refer to page 42):

$$A_a B^a = const \text{ therefore } \nabla_b(A_a B^a) = \partial_b(A_a B^a) = 0 \text{ so}$$

(1) $$\nabla_b A_a\, B^a + A_a\, \nabla_b B^a = 0,$$

(2) $$A_a \partial_b B^a + B^a \partial_b A_a = 0$$

From (1) It follows that

(3) $$B^a \nabla_b A_a = -A_a \nabla_b B^a = -A_a(\partial_b B^a + \Gamma^a_{cb} B^c) = -A_a \partial_b B^a - \Gamma^a_{cb} B^c A_a$$

Considering the last term, we can swap $a \leftrightarrow c$ to change

$$-\Gamma^a_{cb} B^c A_a \rightarrow -\Gamma^c_{ab} B^a A_c$$

Then considering the fisrt term, from (2) we can write

$$-A_a \partial_b B^a = +B^a \partial_b A_a$$

Now by replacing the two terms in (3) we obtain

$$B^a \nabla_b A_a = B^a \partial_b A_a - \Gamma^c_{ab} B^a A_c$$

But the vector B^a being an arbitrary vector we can get rid of it which gives equation(4.11):

$$\nabla_b A_a = \partial_b A_a - \Gamma^c_{ab} A_c$$

4.3. Page 73 formula (4.16) at the bottom should be :

$$\Gamma^a_{\ bc} = \frac{1}{2} g^{ad} \left(\frac{\partial g_{bd}}{\partial x^c} + \frac{\partial g_{cd}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d} \right)$$

4.4. Page 78, we could solve EXAMPLE 4-7 very nicely using (4.5) definition rather than (4.16)

(4) $$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad}$$

$$\partial_c g_{ab} = \partial_c(e_a \cdot e_b) = \frac{\partial e_a}{\partial x^c} e_b + e_a \frac{\partial e_b}{\partial x^c}$$

But from page 64, (4.5) basic definition we know how to express a derivative of basis vector e_n, so the last equation can be written

$$\partial_c g_{ab} = \Gamma^d_{ac}\, e_d \cdot e_b + \Gamma^d_{bc}\, e_d \cdot e_a$$

(5) $$\partial_c g_{ab} = g_{db}\, \Gamma^d_{ac} + g_{da}\, \Gamma^d_{bc}$$

Now replacing the expression (5) into (4), we see that the sum is identically nul and so $\nabla_c g_{ab} = 0$. Therefore from (4) it is easy to retrieve directly EXEMPLE 4-3 that is

$$\partial_c g_{ab} = \Gamma_{abc} + \Gamma_{bac}$$

4.5. Page 80, The exterior Derivative (4.15) equation: $d^2 = 0$, and $d(d\alpha) = 0$. Do you find it obvious ? I do not. Could someone help ?
Is it something like $d^2 f = d\left(\frac{\partial f}{\partial x^i} dx^i\right)$ which will give :

$$\left(\frac{\partial^2 f}{\partial x^i}\right) dx^i \wedge dx^i \quad \text{and as the wedge product is null, yields} \quad d^2 f = 0 \ ?$$

4.6. As stated in page 82, the geodesic equation provides a nice shorcut that can be used to obtain quickly the Chistoffel symbols. Recall that this equation has the form:

$$\frac{d^2 x^a}{ds^2} + \Gamma^a_{\ bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0 \tag{6}$$

Note that summation convention is in effect for the second term of (6), which has not been taken into account in SOLUTION 4-10 (I will give you the right answer in the next paragraph) but most important is the physical meaning of this equation. One can call the quantity $-m\Gamma^a_{\ bc} u^b u^c$ the 4-*force* acting on the particle of masse m in the gravitational field. The metric tensor g_{ab} here plays the part of *potentials* of the gravitational field and its spacial derivatives characterize the *intensity* $\Gamma^a_{\ bc}$ of the field. (which is consistent with $g \approx 1 + \frac{2\phi}{c^2}$ where ϕ is rightly the gravitational potential in newtonian theory- see page 214-).

4.7. Page 84 EXAMPLE 4-10. Let's display the metric tensor and K definition (I was myself confused at first, may be due to too many Greek letters!). Starting from

$$ds^2 = \xi^2 d\tau^2 - d\xi^2 \quad \text{which gives:}$$

$$g_{ab} = \begin{pmatrix} \xi^2 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

$$K = \frac{1}{2}(g_{\tau\tau}\dot{\tau}^2 + g_{\xi\xi}\dot{\xi}^2) = \frac{1}{2}(\xi^2\dot{\tau}^2 - \dot{\xi}^2) \tag{8}$$

Now the Euler/Lagrange equation with respect to τ is :

$$\frac{d}{ds}\left(\frac{\partial K}{\partial \dot{\tau}}\right) = \frac{\partial K}{\partial \tau} \quad \text{which gives:}$$

$$\frac{d}{ds}\left(\xi^2\dot{\tau}\right) = 2\xi\dot{\xi}\dot{\tau} + \xi^2\ddot{\tau} \quad \text{and} \quad \frac{\partial K}{\partial \tau} = 0$$

which is the equation that we have at the end of the exercise

$$\ddot{\tau} + \frac{2}{\xi}\dot{\xi}\dot{\tau} = 0$$

This equation identified with the geodesic equation (do not forget the summation) should be written:

$$\ddot{\tau} + \frac{1}{\xi}\dot{\xi}\dot{\tau} + \frac{1}{\xi}\dot{\tau}\dot{\xi} = 0$$

which yields:

$$\Gamma^\tau_{\ \xi\tau} = \Gamma^\tau_{\ \tau\xi} = \frac{1}{\xi}, \quad \text{not} \quad \frac{2}{\xi}$$

Note that there is no such error in formula (4.40) as the differential equation is:

$$\ddot{\xi} + \xi\dot{\tau}^2 = 0 \quad \text{and so } \Gamma^{\xi}_{\ \tau\tau} = \xi$$

4.8. Riemann tensor Symmetries.

To retrieve and remember the symmetries, use the formula given in the particular context of Cosmology page 257 formula (12.1)

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc})$$

So you could check easily and remember that:

$$R_{abcd} = R_{cdab} = -R_{bacd} = -R_{abdc} = R_{badc}$$

4.9. Quiz 2 (a) I've got a problem with the Christoffel symbol of the first kind :
Using formula (4.15) page 72 yields $\Gamma_{\theta\phi\phi} = r^2 \sin\theta\cos\theta$
But we can do the same calculation starting from the Christoffel symbol of the second kind and using the metric tensor to retrieve the previous result. Using formula (4.16) first and recall that $g_{\theta\theta} = r^2$ and $g_{\phi\phi} = r^2\sin^2\theta$:

$$\Gamma^{\theta}_{\ \phi\phi} = -\frac{1}{2}g^{\theta\theta}\partial_\theta g_{\phi\phi} = -\sin\theta\cos\theta$$

Now using the metric tensor to retrieve the first result:

$$\Gamma_{\theta\phi\phi} = g_{\theta\theta}\Gamma^{\theta}_{\ \phi\phi} = -r^2\sin\theta\cos\theta$$

As you can see I have found the exact opposite!
So I had another go using the geodesic equation (it is a very good and useful exercise anyway):

$$K = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)$$

$$\frac{d}{ds}\left(\frac{\partial K}{\partial\dot{\theta}}\right) = \frac{\partial K}{\partial\theta}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

So by identification we are able to find easily:

$$\Gamma^{\theta}_{\ r\theta} = \Gamma^{\theta}_{\ \theta r} = \frac{1}{r} \quad \text{and also } \Gamma^{\theta}_{\ \phi\phi} = -\sin\theta\cos\theta \quad \text{which is consistent with (4.16).}$$

Conclusion: I do not know why there is this discrepancy in the first place. Anyway the good news is that in practicing *"relativity DeMYSTiFieD"* we do not need to calculate the Christoffel symbols of the first kind but only need the more friendly Christoffel symbols of the second kind (= the one you can recognize at once the symmetry by the two low indices!)

5. Chapter 5

This section and the previous one are very important chapters from a mathematical viewpoint and hence every example should be carefully examined.

5.1. Page 105, about *curvature one forms* index gymnastic the fact that using $\eta_{\hat{a}\hat{b}} = diag(1,-1,-1,-1)$ or $\eta_{\hat{a}\hat{b}} = diag(-1,1,1,1)$ does not change the outcome. In fact the rule is the same as for all antisymmetric tensors, like the Faraday F_{ik} tensor in E.M. albeit is the curvature one forms really a tensor ?
Nevertheless this index gymnastic antisymmetric tensors (like $F_{ik} = F_{ki}$) gives me a real headache, it is difficult to understand why the derived mixed tensor F^k_i is not antisymmetric on the timelike components whereas it is still in its spacelike components ! A similar calculation like $\Gamma^{\hat{a}}_{\ \hat{b}}$ gives :

$$F^k_i = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} \quad \text{(MKSA units)}$$

I had doubt about this result but I was really relieved seeing the same formula in the "Gravitation" book by W. Misner, Kip S. Thorne and J. A. Wheeler, page 73. This cheers me up a great deal, especially because this mixed tensor is used in the calculation of the E.M. energy-momentum components as you will see the energy density component in section 6.1 !

5.2. Page 105 again, to transform from orthonormal to coordinate frame one use the transformation matrices in the formula (5.14) such as:

$$\Gamma^a_{\ bc} = \left(\Lambda^{-1}\right)^a_{\ \hat{d}}\,\Gamma^{\hat{d}}_{\ \hat{e}\hat{f}}\Lambda^{\hat{e}}_{\ b}\Lambda^{\hat{f}}_{\ c} \tag{9}$$

To help remember the formula I give you my mnemonic tip :
$\Lambda^{\hat{e}}_{\ b}$ = diag(positive sqrt(line element)) for covariant coordinate basis, hence b without hat. Contrarywise, $(\Lambda^{-1})^a_{\ \hat{d}}$ = inverse matrix for contravariant coordinate basis, hence a without hat! Well, it's a try!

My understanding from this chapter is that the element $\omega^{\hat{a}}_{\ b}$ of the matrix are defined by the line element $g = \eta_{\hat{a}\hat{b}}\,\omega^{\hat{a}} \otimes \omega^{\hat{b}}$ with $\omega^{\hat{a}} = \omega^{\hat{a}}_{\ b}dx^b$ which I conclude that $\omega^{\hat{a}}_{\ b}$ are all positive because the signs are included in the tensor $\eta_{\hat{a}\hat{b}}$ as given in example 5-1 page 112. I highlight this point as there is a mistake commented in section 6.5.

5.3. Page 114, mistake to be avoided.

In the calculation of $d\Gamma^{\hat{\phi}}_{\ \hat{\theta}} = d(\cot\theta\,\omega^{\hat{\phi}})$ I was tempted to take a shortcut, i.e. to write directly:

$$d(\cot\theta\,\omega^{\hat{\phi}}) = d(\cot\theta) \wedge \omega^{\hat{\phi}} \tag{10}$$

this calculation leads to:

$$d\Gamma^{\hat{\phi}}_{\ \hat{\theta}} = \frac{1}{\sin^2\theta}\omega^{\hat{\phi}} \wedge \omega^{\hat{\theta}}$$

Which is not the right result given in (5.30)! It does not work, I suppose because equality (10) is not allowed. Recall from page 80 equation (4.26)

$$d(f_\alpha dx^\alpha) = df_\alpha \wedge dx^\alpha$$

So you need first to express the total derivative using dx^α because $dx^\alpha \neq \omega^{\hat{\alpha}}$ and in a second step switch from coordinate to orthonormal basis. . .

5.4. In the SOLUTION 5-3 page 118, last formula of (5.33)

$$R^{\hat{\phi}}_{\ \hat{r}\hat{r}\hat{\phi}} = -\frac{\dot{a}^2 + k}{a^2} \quad \text{not} \quad -\frac{\dot{a}^2}{a^2}$$

This result comes from the calculation of $\Omega^{\hat{\phi}}_{\ \hat{r}}$ not given in the book.

5.5. Quiz 2 (a) the answer is $-r\sin^2\theta$ (minus sign missing coming from the Ricci coefficient $-\frac{1}{r}$ in quiz 1 (a) above)

6. Chapter 6

6.1. Page 137 from the Poisson's equation it is easy to get a flavor of the Einstein's equation using the expression of g_{00}:

$$g_{00} \approx 1 + \frac{\phi}{c^2} \rightarrow \nabla^2 g_{00} \approx \frac{\nabla^2 \phi}{c^2} \quad \text{replacing in Poisson's equation :}$$

$$\nabla^2 g_{00} = \frac{4\pi G\rho}{c^2} = \frac{4\pi G}{c^4} c^2 \rho = \frac{4\pi G}{c^4} T_{00} \tag{11}$$

With $T_{00} = c^2\rho$ being the energy density ($Joule/m^3$) which is consistent with the expression of the electro-magnetic Maxwell tensor where $T^{00} = T_{00} = (\epsilon_0 E^2 + \mu_0 H^2)/2$ also expressed in the same units.

Therefore equation (11) shows why it is reasonable to expect a close relationship between the derivatives of g_{ab} and T_{ab}.

6.2. Page 138 the constant k of the Einstein's equation is the reduced form ($c = 1$). In not reduced form k is given by the equation (11).

In vacuum the Einstein equations become $R_{ab} = 0$. Be aware that this equation does not imply that the space-time *is flat*, as for that more stringent conditions should be verified as $R_{abcd} = 0$.

Another consequence of the Einstein Equation is to show that for the EM field, the metric is flat. Starting from the Einstein equation written with mixed tensors :

$$R_i^k - \frac{1}{2}\delta_i^k R = kT_i^k$$

Now setting $k = i$ and taking into account $\delta_i^i = 4$ with $R_i^i = R$ and $T_i^i = T$ we obtain the equation :

$$R = -kT$$

Now considering the EM field, we know that the energy-momentum tensor is traceless, i.e. $T_i^i = 0$, which means that $R = 0$, therefore the space time is flat for the EM field.

6.3. Page 141, example 6-2, last line, last term of the RHS of the equation should have a minus sign.

6.4. Page 147, example 6-3, a sign error, it should be $R^{\hat{\phi}}_{\ \hat{t}\hat{\phi}\hat{r}} = -R^{\hat{t}}_{\ \hat{\phi}\hat{r}\hat{\phi}}$ which explains the change of sign in formula (6.28).

6.5. Page 149, example 6-3 there is a problem here with the transformation matrix. Recall from my comment in 5.2 the matrix elements should be all positive. But here $\Lambda_t^{\hat{t}} = -1$ in this example with the $(-1, 1, 1, 1)$ metric instead of $(1, -1, -1, -1)$ usually. Fortunately except for formula (6.36) where the consequence is a change of sign, there is no other impact in the following formulae. So the transformation matrix shall be:

$$diag[1, e^{b(t,r)}, R(t,r)]$$

Verification: From the example 6-2:

$$\Gamma^{\hat{t}}{}_{\hat{\phi}\hat{\phi}} = \frac{1}{R}\frac{\partial R}{\partial t} \quad \text{and} \quad \Gamma^{\hat{t}}{}_{\hat{r}\hat{r}} = \frac{\partial b}{\partial t}$$

Applying the transformation matrix to find these two Christoffel symbols in coordinate basis, we find:

$$\Gamma^{t}{}_{\phi\phi} = R\frac{\partial R}{\partial t} \quad \text{and} \quad \Gamma^{t}{}_{rr} = \frac{\partial b}{\partial t}e^{2b} \tag{12}$$

Now we can find them directly using the preferred method, I mean the geodesic equation. From the line element the expression of K is :

$$K = \frac{1}{2}\left(-\dot{r}^2 + e^{2b(t,r)}\dot{r}^2 + R^2(t,r)\dot{\phi}^2\right)$$

Now the Euler/Lagrange equation with respect to the time t is such :

$$\frac{d}{ds}\left(\frac{\partial K}{\partial \dot{t}}\right) = \frac{\partial K}{\partial t} \quad \text{so}$$

$$\frac{d}{ds}(-\dot{t}) = -\ddot{t} \quad \text{and} \quad \frac{\partial K}{\partial t} = \frac{1}{2}\left(2e^{b}\frac{\partial b}{\partial t}\dot{r}^2 + 2R\frac{\partial R}{\partial t}\dot{\phi}^2\right)$$

Which gives the following geodesic equation :

$$\ddot{t} + e^{2b}\frac{\partial b}{\partial t}\dot{r}^2 + R\frac{\partial R}{\partial t}\dot{\phi}^2 = 0$$

Therefore by identification we retrieve the two equations stated in line (12) above.

(Nota: I did not mentioned but in the line element $R(t,r)$ is squared of course !)

6.6. Quiz: I have stopped at quiz 6 where my answer is (c) like in the book, however for the quiz 5, I have found answer (b) not (a) i.e. with a minus sign. Because it is a similar calculation the minus sign seems justify for both exercises. What have you found?

7. Chapter 7

For the moment just a comment to say: Regarding the EM tensor T^{ik} the equation $\frac{\partial T^{ik}}{\partial k} = 0$ holds as well for the same reason (energy and momentum conservation). Moreover this tensor is traceless.

8. Chapter 8

The best application of Killing vectors I have seen is the book is for the calculation of the conserved quantities for the Schwarzschild solution starting from page 219.

9. Chapter 9

Nothing to comment, waiting for practical application.

10. Chapter 10

Not finished yet

11. Chapter 11

Not finished yet

12. Chapter 12

Not finished yet

13. Chapter 13

This section is quite challenging, I have not finished yet. To begin with I would like to show the EM case which has some commonalities which the topic here in particular about gauge condition.

Starting with the source equation and considering waves propagating in the vacuum with no electric charge or current i.e. $j^i = 0$ and so:

$$\partial_k F^{ik} = 0 \quad \text{with} \quad F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k}$$

A^i being the quadri-potential of the EM field, equivalent to the g_{ik} tensor in the gravitational field [1]. The derivation gives two terms:

$$\frac{\partial^2 A^k}{\partial x_i \partial x^k} - \frac{\partial^2 A^i}{\partial x_k \partial x^k} = 0$$

Now we can fixed a condition for the potential A^i and imposed $\frac{\partial A^k}{\partial x^k} = 0$ which is called Lorentz conditions or *Lorentz gauge*, which in vector notation and with the usual ϕ and $\vec{A}$ respectively scalar and vector potentials gives:

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} = 0$$

So finally we do have the wave equation for the potential propagation which can be written using the d'Alembertian operator ($\Box$ usually):

$$\frac{\partial^2 A^i}{\partial x_k \partial x^k} = 0 = \Box A = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}\right)$$

As you can see, the calculation in tensorial guise is very concise and straightforward compared to the more convoluted vector calculus demonstration. This is especially true for the Lorentz condition which appears much more "natural" in tensor notation than in vectorial one.

14. Final Exam

[1] because for the EM field the potential is defined with one index tensor (or vector) and for the gravitational field with a two indexes tensor, this explains why the photon has a spin 1 and why the graviton is expected to have a spin = 2 (QTF in a Nutshell from A. Zee).